Culham Plasma Physics Summer School introductory maths

These questions have been chosen to illustrate some of the mathematical techniques used during the Culham Plasma Physics Summer School. They are not compulsory but familiarity with the methods and language will make the lectures easier to follow.

Useful formulae and vector identities (page 3), and worked answers (page 4) can be found following the questions.

1. Prove the vector identities $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ and $\nabla \times \nabla f = \mathbf{0}$. This can be done by writing out the components in full or by using summation convention.

2. The Earth's dipole magnetic field can be approximated by

$$\mathbf{B} = -\nabla \frac{\mu_0 \mu_{\mathbf{E}} \cdot \mathbf{r}}{4\pi r^3}$$

where the Earth's magnetic moment $\mu_{\mathbf{E}} = 8 \times 10^{22}$ A m^2 is a vector pointing along the Earth's axis of rotation and $\mu_0 = 4\pi \times 10^{-7}$ T m A^{-1} is the magnetic constant.

Using spherical polar co-ordinates, derive an expression for the strength of the Earth's magnetic field above the equator as a function of r and hence calculate this value for the surface of the Earth ($R_E = 6.4 \times 10^6$ m) and at $r = 5R_E$.

3. Maxwell's equations for electromagnetism in free space can be written[1]

$$\begin{array}{lr} \text{(i)} & \nabla \cdot \mathbf{B} = 0 \\ \text{(ii)} & \nabla \cdot \mathbf{E} = 0 \\ \text{(iii)} & \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \\ \text{(iv)} & \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0} \end{array}$$

A vector $\mathbf{A}$ is defined by $\mathbf{B} = \nabla \times \mathbf{A}$, and a scalar ϕ by $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$. Show that if the condition

$$\text{(v)} \quad \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

is imposed, then both $\mathbf{A}$ and ϕ satisfy the wave equations

$$\begin{array}{lr} \text{(vi)} & \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \\ \text{(vii)} & \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{0} \end{array}$$

(a) First verify that the expressions for $\mathbf{B}$ and $\mathbf{E}$ in terms of $\mathbf{A}$ and ϕ are consistent with (i) and (iii).

(b) Substitute for $\mathbf{E}$ in (ii) and use the derivative with respect to time of (v) to eliminate $\mathbf{A}$ from the resulting expression. Hence obtain (vi).

[1]This question is taken from *Mathematical Methods for Physics and Engineering*, Riley *et al*, CUP.

(c) Substitute for $\mathbf{B}$ and $\mathbf{E}$ in (iv) in terms of $\mathbf{A}$ and ϕ. Then use the divergence of (v) to simplify the resulting equation and so obtain (vii).

4. Assume a small perturbation of the form $\xi(\mathbf{r}, t) = \xi_0 \exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t))$ occurs in the $\mathbf{E}$ and $\mathbf{B}$ fields in free space. Use equations (iii) and (iv) in Question 3 to derive the dispersion relation for an electromagnetic plane wave in free space.

   (a) First consider what the operators ∇ and $\frac{\partial}{\partial t}$ might look like for such a perturbation.

   (b) Use the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ to eliminate $\mathbf{E}$ and $\mathbf{B}$, and hence find the dispersion relation.

   (c) Write down the phase velocity ($v_p = \frac{\omega}{k}$) and group velocity ($v_g = \frac{\partial \omega}{\partial k}$) of such waves.

5. The first three moments of a distribution function f are given by

   (a) 0^{th}: $n = \int f d^3v$ (number density)

   (b) 1^{st}: $\mathbf{u} = \frac{1}{n} \int \mathbf{v} f d^3v$ (fluid velocity)

   (c) 2^{nd}: $\mathbb{P} = m \int \mathbf{v}\mathbf{v} f d^3v$ (pressure tensor)

   where the integral is over all velocity space and $d^3v = dv_x dv_y dv_z$. The Maxwell-Boltzmann distribution function in a plasma is given by

$$f(\mathbf{r}, \mathbf{v}) = n_0 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{q\phi(\mathbf{r})}{kT}\right) \exp\left(-\frac{m}{2kT}(v_x^2 + v_y^2 + v_z^2)\right)$$

   where n_0 is the overall average number density, m is the particle mass, q is particle charge, $\phi(\mathbf{r})$ is a potential field at position $\mathbf{r}$, k and T have their usual meanings and v_x, v_y and v_z are the components of the particle velocity vector $\mathbf{v}$.

   Find the first three moments of this distribution function.

   (For the pressure tensor, the on-diagonal ("ordinary pressure") terms are given by $P_{xx} = m \int v_x v_x f d^3v$ and so on, and the off-diagonal (viscosity) terms are given by $P_{xy} = m \int v_x v_y f d^3v$ *etc.*)

Useful equations

Integrals

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

Summation convention

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i$$

$$(\mathbf{u} \times \mathbf{v})_i = \varepsilon_{ijk} u_j v_k$$

where ε_{ijk} is the Levi-Citiva symbol: $\varepsilon_{123} = 1$; $\varepsilon_{ijk} = -\varepsilon_{ikj}$.

Vector identities

1. $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{B} \times \mathbf{C} \cdot \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$
2. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
3. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$
4. $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$
5. $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})\mathbf{D}$
6. $\nabla(fg) = \nabla(gf) = f\nabla g + g\nabla f$
7. $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
8. $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A}$
9. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$
10. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
11. $\mathbf{A} \times (\nabla \times \mathbf{B}) = (\nabla\mathbf{B}) \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
12. $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
13. $\nabla^2 f = \nabla \cdot \nabla f$
14. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$
15. $\nabla \cdot (\mathbf{A}\mathbf{B}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B}$

Note on formulae

Many useful plasma physics formulae can be found in the *NRL Plasma Formulary*, which can be downloaded from http://wwwppd.nrl.navy.mil/nrlformulary/

Answers

1. The ∇ operator can be written as

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

and therefore

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix}$$

so

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{A}) &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} \\ &= 0 \end{aligned}$$

and

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

so

$$\begin{aligned} \nabla \times \nabla f &= \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix} \\ &= \mathbf{0} \end{aligned}$$

Using summation convention (and relying on $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$):

$$\begin{aligned} \nabla \times \mathbf{A} &= \varepsilon_{ijk} \nabla_j A_k \\ \nabla \cdot \nabla \times \mathbf{A} &= \nabla_i \sum_{jk} \varepsilon_{ijk} \nabla_j A_k \\ &= \sum_{ijk} \varepsilon_{ijk} \nabla_i \nabla_j A_k \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \nabla f &= \nabla_i f \\ \nabla \times \nabla f &= \varepsilon_{ijk} \nabla_j \nabla_k f \\ &= \mathbf{0} \end{aligned}$$

2. We define a co-ordinate system with the z-axis aligned to the Earth's magnetic moment and the origin at the Earth's core, and then our spherical polar co-ordinate is given by $\mathbf{r} = (r, \theta, \phi)$ where r is the radius, θ is the angle between the magnetic moment and $\mathbf{r}$ (*i.e.* $\theta = \frac{\pi}{2}-$latitude), and ϕ is the angle around the equator (*i.e.* longitude). Then

$$\mu_{\mathbf{E}} \cdot \mathbf{r} = \mu_E r \cos\theta$$

In spherical polar co-ordinates

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi$$

where $\hat{\mathbf{e}}_{\mathbf{r}}, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ are the basis vectors of the spherical co-ordinate system.

Therefore

$$\begin{aligned}\mathbf{B} &= -\nabla\frac{\mu_0\mu_E\cos\theta}{4\pi r^2} \\ &= \frac{\mu_0\mu_E\cos\theta}{2\pi r^3}\hat{\mathbf{e}}_{\mathbf{r}} + \frac{\mu_0\mu_E\sin\theta}{4\pi r^3}\hat{\mathbf{e}}_\theta\end{aligned}$$

At (or above) the equator $\theta = \frac{\pi}{2}$, and so the radial term vanishes and $\sin\theta = 1$. Thus

$$B = \frac{\mu_0\mu_E}{4\pi r^3}$$

and $B(R_E) = 3.05 \times 10^{-5}$ T and $B(5R_E) = 2.44 \times 10^{-7}$ T.

3. (a) Substituting $\mathbf{B} = \nabla \times \mathbf{A}$ into (i) gives $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, which is consistent as the div of a curl is always zero (*cf.* Question 1). Substituting for $\mathbf{E}$ into (iii) and using $\nabla \times \nabla f = \mathbf{0}$ gives

$$\begin{aligned}\mathbf{0} &= \nabla \times (-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{1}{c}\nabla \times \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ &= \mathbf{0}\end{aligned}$$

(b) Taking the derivative with respect to time of (v) gives

$$\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\frac{\partial^2 \phi}{\partial t^2} = 0$$

and substituting for $\mathbf{E}$ into (ii) gives

$$\begin{aligned}0 &= \nabla \cdot (-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) \\ &= -\nabla^2\phi - \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} \\ &= \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}\end{aligned}$$

(c) Noting that $\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, taking the grad of (v) $(\nabla(\nabla \cdot \mathbf{A}) + \frac{1}{c}\nabla\frac{\partial \phi}{\partial t} = 0)$, and substituting these into (iv) gives

$$-\frac{1}{c}\nabla\frac{\partial \phi}{\partial t} - \nabla^2 \mathbf{A} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$$

Taking the derivative with respect to time of $\mathbf{E}$ gives

$$\frac{\partial \mathbf{E}}{\partial t} = -\nabla\frac{\partial \phi}{\partial t} - \frac{1}{c}\frac{\partial^2 \mathbf{A}}{\partial t^2}$$

and substituting this into the previous equation gives

$$\begin{aligned}
\mathbf{0} &= -\frac{1}{c}\nabla\frac{\partial \phi}{\partial t} - \nabla^2 \mathbf{A} - \frac{1}{c}(-\nabla\frac{\partial \phi}{\partial t} - \frac{1}{c}\frac{\partial^2 \mathbf{A}}{\partial t^2}) \\
&= -\frac{1}{c}\nabla\frac{\partial \phi}{\partial t} + \frac{1}{c}\nabla\frac{\partial \phi}{\partial t} - \nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \\
&= \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}
\end{aligned}$$

4. (a) $\frac{\partial}{\partial t}\xi_0 \exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t)) = -i\omega\xi_0 \exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t)) = -i\omega\xi$, and thus $\frac{\partial}{\partial t} = -i\omega$. For the ∇ operator, $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ and we can consider each term individually:

$$\begin{aligned}
\nabla_x \xi &= \frac{\partial}{\partial x}\xi_0 \exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t)) \\
&= \xi_0 \exp(i(k_y y + k_z z - \omega t))\frac{\partial}{\partial x}\exp(ik_x x) \\
&= \xi_0 \exp(i(k_y y + k_z z - \omega t)) ik_x \exp(ik_x x) \\
&= ik_x \xi
\end{aligned}$$

and similarly with ∇_y and ∇_z, so $\nabla = ik_x\mathbf{i} + ik_y\mathbf{j} + ik_z\mathbf{k} = i\mathbf{k}$.

(b) From equation (iii) (and substituting for equation (iv) at the appropriate point):

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
i\mathbf{k} \times \mathbf{E} &= \frac{1}{c}i\omega\mathbf{B} \\
i\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) &= \frac{\omega}{c}i\mathbf{k} \times \mathbf{B} \\
&= \frac{\omega}{c}\frac{-i\omega}{c}\mathbf{E} \\
\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - \mathbf{E}(\mathbf{k}\cdot\mathbf{k}) &= -\frac{\omega^2}{c^2}\mathbf{E} \\
k^2\mathbf{E} &= \frac{\omega^2}{c^2}\mathbf{E}
\end{aligned}$$

as $\mathbf{k}\cdot\mathbf{E}=0$ for a plane wave (the displacement in $\mathbf{E}$ is perpendicular to the wavevector $\mathbf{k}$). Dividing by $\mathbf{E}$ gives the dispersion relation

$$\omega^2 = c^2k^2$$

(c) The phase velocity $v_p = \frac{\omega}{k} = c$, and the group velocity $v_g = \frac{\partial\omega}{\partial k} = c$ as well.

5. First, define $\alpha = \frac{m}{2kT}$.

(a) 0^{th} moment: $n = \int f d^3v$

$$\begin{aligned}
n &= \int f d^3v \\
&= n_0\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{q\phi(\mathbf{r})}{kT}\right)\int\int\int \exp\left(-\alpha(v_x^2+v_y^2+v_z^2)\right) dv_x dv_y dv_z \\
&= n_0\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{q\phi(\mathbf{r})}{kT}\right)\int\int \exp\left(-\alpha(v_y^2+v_z^2)\right)\int \exp\left(-\alpha(v_x^2)\right) dv_x dv_y dv_z \\
&= n_0\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{q\phi(\mathbf{r})}{kT}\right)\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\int\int \exp\left(-\alpha(v_y^2+v_z^2)\right) dv_y dv_z \\
&= n_0\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{q\phi(\mathbf{r})}{kT}\right)\left(\frac{\pi}{\alpha}\right)^{\frac{3}{2}} \\
&= n_0 \exp\left(-\frac{q\phi(\mathbf{r})}{kT}\right)
\end{aligned}$$

(b) 1^{st} moment: $\mathbf{u} = \frac{1}{n}\int \mathbf{v} f d^3v$

Consider just the x component of the velocity:

$$\begin{aligned}
u_x &= \frac{1}{n}\int v_x f d^3v \\
&= \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int\int\int v_x \exp\left(-\alpha(v_x^2+v_y^2+v_z^2)\right) dv_x dv_y dv_z \\
&= \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int\int \exp\left(-\alpha(v_y^2+v_z^2)\right)\int v_x \exp\left(-\alpha v_x^2\right) dv_x dv_y dv_z
\end{aligned}$$

This integral ($\int v_x \exp\left(-\alpha v_x^2\right) dv_x$) is an odd function and therefore equals zero. Thus $u_x = 0$, and similarly $u_y = u_z = 0$ as well.

(c) 2^{nd} moment: $\mathbb{P} = m\int \mathbf{v}\mathbf{v} f d^3v$

First consider off-diagonal terms:

$$\begin{aligned}
P_{xy} &= m\int v_x v_y f d^3v \\
&= mn\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int\int\int v_x v_y \exp\left(-\alpha(v_x^2+v_y^2+v_z^2)\right) dv_x dv_y dv_z \\
&= mn\left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}}\int\int v_y \exp\left(-\alpha(v_y^2+v_z^2)\right)\int v_x \exp\left(-\alpha v_x^2\right) dv_x dv_y dv_z
\end{aligned}$$

Once again $\int v_x \exp\left(-\alpha v_x^2\right) dv_x = 0$ and so $P_{xy} = 0$. Similarly, all the other off-diagonal terms equal zero.

On-diagonal terms:

$$
\begin{aligned}
P_{xx} &= m \int v_x v_x f d^3 v \\
&= mn \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \int\int\int v_x v_x \exp\left(-\alpha(v_x^2 + v_y^2 + v_z^2)\right) dv_x dv_y dv_z \\
&= mn \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \int\int \exp\left(-\alpha(v_y^2 + v_z^2)\right) \int v_x^2 \exp\left(-\alpha v_x^2\right) dv_x dv_y dv_z \\
&= mn \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \int\int \exp\left(-\alpha(v_y^2 + v_z^2)\right) dv_y dv_z \\
&= mn \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \left(\frac{\pi}{\alpha}\right) \\
&= \frac{mn}{2\alpha} \\
&= nkT
\end{aligned}
$$

and similarly for P_{yy} and P_{zz}. So the pressure tensor for a Maxwellian distribution function is

$$
\mathbb{P} = \begin{pmatrix} nkT & 0 & 0 \\ 0 & nkT & 0 \\ 0 & 0 & nkT \end{pmatrix}
$$